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A two-parameter friction model☆

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ABSTRACT

The new friction model proposed in this paper takes all types of friction into account: sliding, pivoting and rolling friction. The model depends on two parameters. With a zero value of one parameter it is converted into the Contensou–Zhuravlev model, and with a zero value of the other parameter it is converted into the Coulomb model.

The interaction of a body with the bearing surface during translational motion of the body is described fairly adequately by the classical model of dry friction (Coulomb's law). In the case of plane-parallel translational motion of the body, the Contensou–Zhuravlev model must be used;[1,2] this model takes both sliding friction and pivoting friction into account. The friction model proposed below is suitable for describing arbitrary translational motion of the body.

The friction model proposed in this paper will be illustrated for the problem of the motion of a homogeneous sphere of radius a along a stationary horizontal plane. Following Contensou,[1] we will replace the point of contact of the sphere with the plane with a contact patch, assuming, unlike Contensou, that it has the shape of a spherical segment[3] rather than of a circle. In other words, we will assume that, when the sphere comes into contact with the plane, both the sphere and the bearing surface undergo elastic deformations. These deformations determine the two parameters of the contact patch: $\delta = a/R \in [0,1]$ and $\varepsilon = r/a \in [0,1]$, where R is the radius of the sphere, a segment of which determines the contact patch, and r is the radius of this segment (see Fig. 1, where a side view of the contact patch is shown on the left, and a top view on the right). Note that such a spherical segment approximates a segment $x^2 + y^2 \le r^2$ of a paraboloid $z = (x^2 + y^2)/(2R)$ up to terms $\varepsilon^4\delta^2$. The case $\delta = 0$ ($R = \infty$) corresponds to the Contensou–Zhuravlev (CZh) model, in which the bearing surface is assumed to be non-deformable, and the case $\varepsilon = 0$ ($r = 0$) corresponds to the Coulomb model, in which the sphere is assumed to be non-deformable. The case $\delta = 1$ ($R = a$) also corresponds to the assumption of sphere non-deformability but requires a separate analysis because it is not described by the Coulomb model.

Let P be an arbitrary point of the contact patch (a spherical segment Σ), and let its position be defined by the angles $\alpha \in [0, 2\pi]$ and $\beta \in [0, \beta_0]$, where $\beta_0 = \arcsin \mu$, $\mu = r/R = \varepsilon\delta$ (see Fig. 1, in which the point O is the centre of the sphere, a segment of which is Σ, and S and C are the centre of the sphere and its projection onto Σ).

We will introduce an orthonormalized reference frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ such that the unit vector $\mathbf{e}_1$ is directed along the sliding velocity $\mathbf{u} = u\mathbf{e}_1$ of the sphere (the velocity of point C of the sphere), the unit vector $\mathbf{e}_2$ is orthogonal to the sliding velocity $\mathbf{u}$ and, like $\mathbf{e}_1$, lies in the horizontal plane and the unit vector $\mathbf{e}_3$ is directed along a rising vertical. We will find the velocity of the point P from Euler's formula

$$\mathbf{v}_P = \mathbf{v}_C + [\boldsymbol{\omega}, \boldsymbol{\rho}]; \quad \mathbf{v}_C = \mathbf{u} = u\mathbf{e}_1, \ \boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3$$

where $\boldsymbol{\omega}$ is the angular velocity of the sphere and $\boldsymbol{\rho}$ is the radius vector of point P relative to the centre C of the contact patch:

$$\rho = \rho_1\mathbf{e}_1 + \rho_2\mathbf{e}_2 + \rho_3\mathbf{e}_3; \ \rho_1 = R\sin\beta\cos\alpha, \rho_2 = R\sin\beta\sin\alpha, \rho_3 = R(1 - \cos\beta)$$

Thus

$$\mathbf{v}_P = \upsilon_1\mathbf{e}_1 + \upsilon_2\mathbf{e}_2 + \upsilon_3\mathbf{e}_3$$

$$\upsilon_1 = u + R\omega_2(1 - \cos\beta) - R\omega_3\sin\beta\sin\alpha$$

$$\upsilon_2 = -R\omega_1(1 - \cos\beta) + R\omega_3\sin\beta\cos\alpha$$

$$\upsilon_3 = R\sin\beta(\omega_1\sin\alpha - \omega_2\cos\alpha)$$

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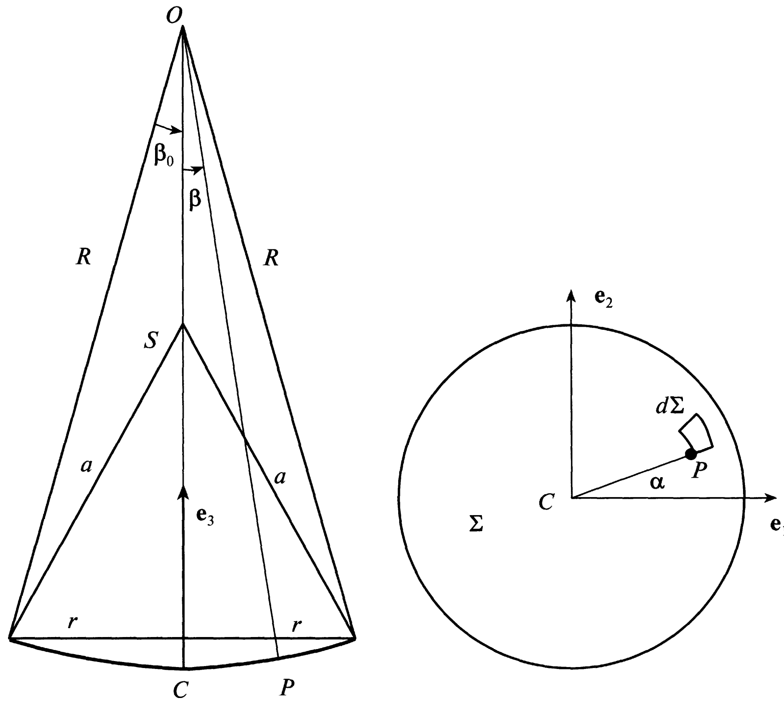

Fig. 1.

We will find the sliding velocity $\mathbf{u}_P$ of the point P, bearing in mind that this point is sliding over the spherical segment Σ (rather than over a horizontal plane, as in the CZh model). For this, we will decompose the velocity of the point P into two components – the tangential component and the normal component to Σ at point P:

$$\mathbf{v}_P = \mathbf{u}_P + (\mathbf{v}_P, \mathbf{n}_P)\mathbf{n}_P; \quad \mathbf{n}_P = -\sin\beta\cos\alpha\mathbf{e}_1 - \sin\beta\sin\alpha\mathbf{e}_2 + \cos\beta\mathbf{e}_3$$

where $\mathbf{n}_P$ is the unit vector of the normal to the segment Σ at the point P, directed towards the concavity of this segment.

Thus

$$\begin{aligned}
&\mathbf{u}_P = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 \\
&u_1 = u(1-\sin^2\beta\cos^2\alpha) + R\omega_1\sin^2\beta\sin\alpha\cos\alpha + \\
&+ R\omega_2(1-\cos\beta-\sin^2\beta\cos^2\alpha) - R\omega_3\sin\beta\sin\alpha \\
&u_2 = -u\sin^2\beta\sin\alpha\cos\alpha - R\omega_1(1-\cos\beta-\sin^2\beta\sin^2\alpha) - \\
&- R\omega_2\sin^2\beta\sin\alpha\cos\alpha + R\omega_3\sin\beta\cos\alpha \\
&u_3 = u\sin\beta\cos\beta\cos\alpha + R\sin\beta(1-\cos\beta)(\omega_1\sin\alpha - \omega_2\cos\alpha)
\end{aligned} \tag{1}$$

According to Hertz's theory, the normal pressure density of the sphere at the point P is given by the formula

$$\nu(P) = \frac{3N}{2\sigma}\sqrt{1-\frac{\rho^2}{\rho_0^2}}; \quad \sigma = 2\pi R^2(1-\cos\beta_0), \quad \rho = 2R\sin\frac{\beta}{2}, \quad \rho_0 = 2R\sin\frac{\beta_0}{2}$$

where N is the normal pressure, σ is the contact patch area, ρ is the distance from the point C to the point P and ρ_0 is the distance from the point C to the contact patch boundary. The pressure of the sphere on an elementary area $d\Sigma$ of the contact patch, constructed at the point P, is given by the formula $N(P) = \nu(P)d\sigma$, where $d\sigma = R^2_{\sin}\ \beta d\beta d\alpha$ is the area $d\Sigma$. Thus

$$N(P) = \frac{3N(\cos\beta-\cos\beta_0)^{1/2}}{4\pi(1-\cos\beta_0)^{3/2}}\sin\beta d\beta d\alpha$$

Assuming that the sliding friction force $d\mathbf{F}(P)$ acting on the elementary area $d\Sigma$ and applied at the point P satisfies Coulomb's law, we have

$$d\mathbf{F}(P) = -kN(P)\mathbf{u}_P/u_P; \quad u_P = (u_1^2 + u_2^2 + u_3^2)^{1/2}$$

where $k > 0$ is the friction coefficient.

The resulting friction force acting on the sphere and applied at the centre of the contact patch (at the point C) is given by the formula

$$\mathbf{F} = \int_\Sigma d\mathbf{F}(P) = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 \tag{2}$$

$$F_i = -k\frac{3N}{4\pi(1-\cos\beta_0)^{3/2}}\int\limits_0^{\beta_0}(\cos\beta-\cos\beta_0)^{1/2}\sin\beta d\beta\int\limits_0^{2\pi}\frac{u_i}{u_P}d\alpha,\ i=1,2$$

Similarly, the principal moment of the friction forces about the point C is given by the formula

$$\mathbf{M_C} = \int\limits_\Sigma[\rho, d\mathbf{F}(P)] = M_1\mathbf{e}_1 + M_2\mathbf{e}_2 + M_3\mathbf{e}_3 \tag{3}$$

$$M_j = -k\frac{3N}{4\pi(1-\cos\beta_0)^{3/2}}\int\limits_0^{\beta_0}(\cos\beta-\cos\beta_0)^{1/2}\sin\beta d\beta\int\limits_0^{2\pi}\frac{w_j}{u_P}d\alpha,\ j=1,2,3$$

$$w_1 = R[u\sin^2\beta\sin\alpha\cos\alpha + R\omega_1(1-\cos\beta)^2 - R\omega_3(1-\cos\beta)\sin\beta\cos\alpha]$$

$$w_2 = R[u(1-\cos\beta-\sin^2\beta\cos^2\alpha) + R\omega_2(1-\cos\beta)^2 - R\omega_3(1-\cos\beta)\sin\beta\sin\alpha]$$

$$w_3 = R\sin\beta[-u\sin\alpha - R(1-\cos\beta)(\omega_1\cos\alpha + \omega_2\sin\alpha) + R\omega_3\sin\beta]$$

Bearing in mind that $R = a/\delta$ and $\sin\beta_0 = \varepsilon\delta$, and taking relations (1)–(3) into account, we conclude that the resulting friction force **F** and the principal moment $\mathbf{M}_C$ of the friction forces about the point C depend on the parameters of the contact patch:

$$\mathbf{F} = \mathbf{F}(\delta,\varepsilon), \mathbf{M}_C = \mathbf{M}_C(\delta,\varepsilon) \tag{4}$$

Here, the moment $\mathbf{M}_C$ depends continuously on these parameters, while the force **F** depends continuously on the parameter δ and discontinuously[2] on the parameter ε (in the vicinity of the zero value of ε).

Furthermore, from relations (2) and (3) it follows that

$$\mathbf{F}(\delta,0) = -kN\mathbf{e}_1,\ \mathbf{M}_C(\delta,0) = 0 \tag{5}$$

$$\mathbf{F}(0,\varepsilon) = F_1^{(0)}\mathbf{e}_1,\ \mathbf{M}_C(0,\varepsilon) = M_3^{(0)}\mathbf{e}_3 \tag{6}$$

Here[1,2]

$$F_1^{(0)} = -k\frac{3N}{2\pi}\int\limits_0^1 s\sqrt{1-s^2}ds\int\limits_0^{2\pi}\frac{u - r\omega_3 s\sin\alpha}{u_0}d\alpha$$

$$M_3^{(0)} = -k\frac{3Nr}{2\pi}\int\limits_0^1 s^2\sqrt{1-s^2}ds\int\limits_0^{2\pi}\frac{r\omega_3 s - u\sin\alpha}{u_0}d\alpha \tag{7}$$

$$r = a\varepsilon,\ u_0 = (u^2 - 2ur\omega_3 s\sin\alpha + r^2\omega_3^2 s^2)^{1/2}$$

where u_0 is the value of the modulus of the sliding velocity of the point P when $\delta = 0$.

Thus, in the case of a point contact of the sphere with the bearing surface, the model proposed reduces to the Coulomb model (5) (we recall that the limit $\mathbf{F}(\delta, \varepsilon)$ does not exist when $\varepsilon \to +0$), and in the case of a plane contact patch it is converted into the CZh model (6).

In the general case, the resulting friction force has a component perpendicular to the sliding velocity of the sphere ($F_2 \neq 0$), while the principal moment of the friction forces about the centre of the contact patch has horizontal components both along the sliding velocity ($M_1 \neq 0$) and perpendicular to it ($M_2 \neq 0$). The first property of the proposed model ($F_2 \neq 0$) can also be obtained in the CZh model if it is assumed that the centre of pressure does not coincide with the centre of the contact patch.[4] It seems that the second property ($M_1 \neq 0$, $M_2 \neq 0$) cannot be obtained within the framework of this model.

Note also that the proposed model of friction, unlike the CZh model, possesses complete dissipation:

$$(\mathbf{F},\mathbf{u}) + (\mathbf{M}_C,\omega) < 0$$

for any **u** and $\boldsymbol{\omega}$ not equal to zero simultaneously.

In particular, if the sphere rolls on the plane without sliding and pivoting, its energy dissipates in the proposed model, while in model (6), (7) it is conserved.

It is curious that the parameter δ occurs in expressions (4) only in a product with the parameter ε, i.e., relations (4) can be represented in the form ($\mu = \varepsilon\delta$)

$$\mathbf{F} = \mathbf{F}(\mu,\varepsilon),\ \mathbf{M}_C = \mathbf{M}_C(\mu,\varepsilon) \tag{8}$$

where the force **F** depends continuously on the on parameter μ. Here, the parameter ε occurs in expressions (8) only in a product with the radius of the sphere a, i.e., relations (8) can be represented in the form ($r = a\varepsilon$ is the radius of the contact patch)

$$\mathbf{F} = \mathbf{F}(\mu,r),\ \mathbf{M}_C = \mathbf{M}_C(\mu,r) \tag{9}$$

Obviously, the moment $\mathbf{M}_C$ depends continuously on μ and r, while the force **F** depends continuously on μ and discontinuously on r (in the vicinity of a zero value of r).

Furthermore, the radius of the contact patch r occurs in the expression for the friction force $\mathbf{F}$ only in a product with the angular velocity $\boldsymbol{\omega}$ of the sphere, i.e., $\mathbf{F}$ depends on the sliding velocity $\mathbf{u}$, the vector $\boldsymbol{\Omega} = r\boldsymbol{\omega}$ and the parameter μ:

$$\mathbf{F} = \mathbf{F}(\mathbf{u}, \Omega, \mu) \tag{10}$$

Here, the moment of the frictional forces can be represented in the form

$$\mathbf{M}_C = r\Phi(\mathbf{u}, \Omega, \mu) \tag{11}$$

Assuming that $\mu \ll 1$, we will expand the components F_i ($i = 1, 2$) of the friction force (10) and M_j ($j = 1, 2, 3$) of the moment of the friction forces (11) in a power series in μ:

$$F_i = F_i^{(0)} + \mu F_i^{(1)} + \mu^2 F_i^{(2)} + O(\mu^3), \quad M_j = M_j^{(0)} + \mu M_j^{(1)} + \mu^2 M_j^{(2)} + O(\mu^3) \tag{12}$$

The components $F_1^{(0)}$ and $M_3^{(0)}$ are given by expressions (7), and the remaining terms of expansions (12) were calculated earlier.[5] In particular, $F_2^{(0)} = 0$ and $M_1^{(0)} = M_1^{(1)} = M_2^{(0)} = 0$; the non-zero expressions for $F_i^{(k)}$ and $M_j^{(k)}$ ($k = 1, 2$) are fairly lengthy[5] and are not given here.

Note that relations (12) hold both for $\delta < 1$ and for $\delta = 1$ (an absolutely rigid sphere on an elastic plane). Thus, taking into account the influence of the elastic properties of the bearing plane ($\delta \neq 0$) on the dynamics of the sphere, we can take the Contensou–Zhuravlev model as the original model of friction (the zero approximation) both in the case of an elastic sphere ($\delta < 1$) and in the case of an absolutely solid sphere ($\delta = 1$). In both cases, in the first approximation, the horizontal components of the force and moment of the friction, which are orthogonal to the sliding velocities of the sphere, must be taken into consideration, and, in the second approximation, the component of the moment of the friction parallel to the sliding velocity must also be considered.

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